Journal of Statistical Physics, Vol. 62, Nos. 3/4, 1991

Diffusive Energy Growth in Classical and Quantum Driven Oscillators

L. Bunimovich,[1,2] H. R. Jauslin,[1,3] J. L. Lebowitz,[1,4] A. Pellegrinotti,[1,5] and P. Nielaba[1,6]

Received June 19, 1990

We study the long-time stability of oscillators driven by time-dependent forces originating from dynamical systems with varying degrees of randomness. The asymptotic energy growth is related to ergodic properties of the dynamical system: when the autocorrelation of the force decays sufficiently fast one typically obtains linear diffusive growth of the energy. For a system with good mixing properties we obtain a stronger result in the form of a central limit theorem. If the autocorrelation decays slowly or does not decay, the behavior can depend on subtle properties of the particular model. We study this dependence in detail for a family of quasiperiodic forces. The solution involves the analysis of a small-denominator problem that can be treated by fairly elementary methods. In the special case of a periodic force the quantum stability problem can be expressed in terms of spectral properties of the Floquet operator. In the presence of resonances the spectrum is absolutely continuous. We find explicitly the eigenvalues and eigenfunctions for the nonresonant case.

KEY WORDS: Time dependent Hamiltonian; diffusive energy growth; quantum chaos; harmonic oscillator.

1. INTRODUCTION

A system subjected to external forces over which it has little or no influence is generally described by a time-dependent Hamiltonian

$$H(t) = H_0(x) + V(x, t) \tag{1.1}$$

[1] Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903.
[2] Shirshov Institute of Oceanology of the Academy of Sciences, Moscow 117218, USSR.
[3] Département de Physique Théorique, Université de Génève, 1211 Geneva 4, Switzerland.
[4] Department of Physics, Rutgers University, New Brunswick, New Jersey 08903.
[5] Dipartimento di Matematica, Università di Roma "La Sapienza," 001187 Rome, Italy.
[6] Institut für Physik, Universität Mainz, D-6500 Mainz, Germany.

0022-4715/91/0200-0793$06.50/0

where H_0 is the Hamiltonian of the isolated system. The effect of the perturbation V on the time evolution of the system will depend on many factors. For microscopic systems such as atoms or molecules, V can serve as an energy pump leading to ionization or dissociation. This can occur either due to strong resonances between the spectrum of $V(t)$ and the natural frequencies of H_0, or to more subtle instabilities which lead to "chaotic behavior." Thus, the multiphoton ionization experiments on Rydberg atoms in a microwave field carried out by Bayfield and Koch[1] were explained by Leopold and Percival[2] and others[3] in terms of chaotic dynamics. These analyses were initially done entirely classically. Their success spurred interest in the study of "quantum chaos," which for our purposes here will simply mean the unstable evolution of the system described by (1.1) when it is treated quantum mechanically. In addition to the ionization of Rydberg atoms there are a variety of other microscopic systems which involve Hamiltonians of the form (1.1). These include ions localized in a Paul or in a Penning trap,[4,5] electrons attracted to a surface of liquid helium,[6] and charge transfer in atomic collisions.[7] Mesoscopic quantum devices, in which quantum coherence plays an essential role, such as small conducting rings threaded by time-dependent magnetic fluxes,[8] also fall within this category.

The problem of stability may be formulated as follows: given a system initially in a state which is localized in "phase space," does the time evolution under (1.1) lead to delocalization?[9] This would correspond in some cases, such as the kicked rotator, to an unbounded growth of the energy, and in others, such as the Rydberg atoms, to ionization.

The case of time-periodic external force has been studied most extensively both for classical and quantum systems. The stability problem can then be expressed in terms of the properties of the Floquet operator $U(t+T, t)$, which gives the evolution of the system over one period T. Classically, U is a canonical volume-preserving Poincaré map in phase space, while quantum mechanically it is a unitary operator on the Hilbert space $\mathscr{H}$ of the unperturbed system. The long-time behavior of the system and its stability are determined by iteration of this map. Classically, the system can be studied by standard methods, including direct numerical simulation. It can exhibit varieties of behavior, ranging from integrable to chaotic, i.e., positive Lyapunov exponents. On the quantum mechanical side, the question of stability can be expressed in terms of the spectral properties of the Floquet operator.[10-13] The energy remains bounded if the spectrum is pure point, and it grows unbounded if the spectrum is continuous. The quantum problem is considerably more difficult than the classical one, and one of the outstanding questions in the field is the existence of qualitatively different behavior of quantum and classical systems. There are models, such as the kicked rotator, which classically show a linear growth of the energy while the quantum analogues have been found to saturate (quantum limitation of diffusion[14,15]).

There has been much progress in solving the time-periodic problem in some special cases. For Hamiltonians that are quadratic in p and q with time-periodic coefficients, the possible asymptotic behaviors have been completely classified,[16] and the spectrum was related to properties of the corresponding classical model. In particular, models describing ions in Paul or in Penning traps (involving parametrically driven quadratic potentials and $1/|x|^2$ interactions[4,5]) have been studied in detail. In addition, the spectrum of the Floquet operator has been proven to be pure point for the smoothly kicked rotator with small coupling.[17] The quantized Fermi accelerator has been shown to have no absolutely continuous spectrum.[18] Some general criteria for stability of the point spectrum have been developed in refs. 19 and 20.

The stability problem is far less understood in the case of nonperiodic perturbations. A generalization of the Floquet theory connecting spectral and stability properties is not available (see, however, ref. 17, where a natural extension was proposed). Existing studies indicate that the quantum limitation of diffusion is weaker or absent for nonperiodic perturbations. This has been observed numerically for quasiperiodic perturbations of the kicked rotator, with two or three incommensurate frequencies.[21] For the randomly kicked rotator, it has been shown that the energy grows unbounded.[22] A general class of randomly perturbed quantum systems in which the time dependence is given by a Markov process has been treated in detail in refs. 23 and 24. It appears from that analysis that quantum systems may be even more unstable than classical systems under such random perturbations. It has also been shown that an arbitrary small dissipation restores the diffusive behavior.[25]

A natural framework for considering general time-dependent perturbations which includes both the periodic and the random potentials as special cases is to write (1.1) in the form

$$H(t) = H_0(x) + V(x, \xi_t) \tag{1.2}$$

where $\xi_t \in \Omega$ is a trajectory of a classical dynamical syustem on a domain Ω, with an invariant ergodic measure μ. One considers then typical or averaged behavior with respect to μ. The time evolution given by (1.2) can be thought of as a limiting case of the system being in contact with an external bath when the relevant state of the bath is described by ξ_t, which is independent of the state of the system. In this language the periodic case corresponds to Ω being the circle with $\xi_t = \xi + \omega t$, and $d\mu = d\xi/2\pi$.

In the present paper we investigate such general time dependence of $H(t)$ in the context of a harmonic oscillator subjected to an external force. The simplicity of the model, in which the classical and quantum behavior are similar, permits us to establish some general relations between the ergodic properties of the dynamical system ξ_t and the growth of the energy of the oscillator. We find that in general when the autocorrelation of the force decays fast enough, the system behaves chaotically and the energy grows linearly. In fact we show more. Analyzing in detail an explicit example with good ergodic properties, we prove that the energy behaves like the square of a Gaussian random variable with variance proportional to time. We expect that this type of behavior occurs whenever ξ_t has a positive Lyapunov exponent. When the correlations do not decay rapidly, the situation is more complicated. The study of some examples indicates that the asymptotic behavior depends on the fine details of the particular model. An interesting example with quasiperiodic correlation is analyzed in Section 4. We find that the asymptotic behavior depends crucially on the smoothness properties of the potential $V(x, \xi_t)$. In Section 5 we discuss the relation between the classical and quantum systems and calculate explicitly the spectrum and eigenfunctions of the Floquet operator for the case of periodic forces.

2. DESCRIPTION OF THE MODELS

The system is described by the Hamiltonian

$$H = \tfrac{1}{2} p^2 + \tfrac{1}{2} \omega_0^2 q^2 - qF(\xi_t) \tag{2.1}$$

where ξ_t is a stationary process with a measure μ, e.g., $F(\xi_t) = A\cos(\omega t + \varphi)$ with φ uniformly distributed in $[0, 2\pi]$. In general

$$\begin{aligned} \langle F(\xi_t)\rangle_\mu &= \langle F(\xi)\rangle_\mu = 0 \\ \langle F(\xi_s)\, F(\xi_t)\rangle_\mu &= \langle F(\xi_{t-s})\, F(\xi)\rangle_\mu \equiv V(|t-s|) \end{aligned} \tag{2.2}$$

The time evolution of the position q_t and momentum p_t for the classical system can be written as

$$v_t \equiv (q_t, p_t/\omega_0) = R_t v_0 + R_t z(t) \tag{2.3}$$

where

$$R_t \doteq \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix} \tag{2.4}$$

and [writing $F(t)$ instead of $F(\xi_t)$ to simplify the notation]

$$z(t)=\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}=\begin{pmatrix} -\int_0^t ds\, F(s)\,\frac{1}{\omega_0}\sin\omega_0 s \\ \int_0^t ds\, F(s)\,\frac{1}{\omega_0}\cos\omega_0 s \end{pmatrix} \tag{2.5}$$

The growth of the energy is determined by the behavior of $z(t)$. Using the Fourier representation

$$F(t)=\omega_0\int_{-\infty}^{\infty} d\omega\, [\tilde{F}_c(\omega)\cos\omega t+\tilde{F}_s(\omega)\sin\omega t] \tag{2.6}$$

(where $\tilde{F}_c$, $\tilde{F}_s$ will in general be distributions), we can write

$$\begin{aligned} z_1(t) &= -\int_{-\infty}^{\infty} d\omega\left[\tilde{F}_c(\omega)\,\frac{1-\cos(\omega_0-\omega)t}{\omega_0-\omega}+\tilde{F}_s(\omega)\,\frac{\sin(\omega_0-\omega)t}{\omega_0-\omega}\right] \\ z_2(t) &= \int_{-\infty}^{\infty} d\omega\left[\tilde{F}_c(\omega)\,\frac{\sin(\omega_0-\omega)t}{\omega_0-\omega}+\tilde{F}_s(\omega)\,\frac{1-\cos(\omega_0-\omega)t}{\omega_0-\omega}\right] \end{aligned} \tag{2.7}$$

From this representation one sees immediately that:

(a) If $\tilde{F}_c(\omega)$ or $\tilde{F}_s(\omega)$ has a component $\sim\delta(\omega-\omega_0)$, there is a resonance and $z(t)\sim t$, i.e., the energy grows quadratically: $E(t)\sim t^2$.

(b) If $\tilde{F}_c(\omega)$ and $\tilde{F}_s(\omega)$ are bounded in a neighborhood of ω_0, then the energy stays bounded, since

$$\int_{-\varepsilon}^{\varepsilon} d\theta\,\frac{\sin\theta t}{\theta}=2\int_0^{\varepsilon t} d\theta'\,\frac{\sin\theta'}{\theta'}\xrightarrow[t\to\infty]{}\text{const}<\infty \tag{2.8}$$

A nontrivial, e.g., linear, growth of the energy is only possible if $\tilde{F}(\omega)$ satisfies neither (a) nor (b). This cannot happen if the force is periodic or quasiperiodic with a Fourier spectrum that is not dense. In such cases the energy is either bounded or grows quadratically. The point of our study is to analyze interesting examples of forces for which the energy grows linearly (or slower) in time—a behavior that is expected to be typical in realistic systems.

We will study the time evolution of the energy of the oscillator averaged over the stationary process μ. It can be written as

$$\begin{aligned} E^c(q_0,p_0,t)=\langle E_t\rangle_\mu &= \left\langle\frac{1}{2}p_t^2+\frac{\omega_0^2}{2}q_t^2\right\rangle_\mu \\ &= \frac{\omega_0^2}{2}\left[q_0^2+\frac{1}{\omega_0^2}p_0^2+\langle|z(t)|^2\rangle_\mu\right] \end{aligned} \tag{2.9}$$

The quantity determining the asymptotic growth is thus $\langle |z(t)|^2 \rangle_\mu$. Using the definition (2.5) and performing one integration (after a change into relative variables), it can be expressed as

$$\omega_0^2 \langle |z_t|^2 \rangle_\mu = \int_{-t}^{t} ds\, (t - |s|)\, C(s) \cos \omega_0 s \tag{2.10}$$

The asymptotic behavior of the averaged energy is thus determined by the correlation function of the force. In particular, if the correlation function decays sufficiently fast, $C(t) = o(t^{-2-\varepsilon})$, then generally the energy grows linearly in time (see Propositions 3.1 and 3.2). If, on the other hand, the correlation function decays more slowly (or does not decay at all), the behavior of the energy can depend on quite subtle details of the force and it is difficult to make general statements. We will discuss examples of both types of behavior in Sections 3 and 4. These examples involve piecewise constant forces that are constructed as follows.

Piecewise Constant Forces

Consider a discrete-time dynamical system on a domain M, defined by a map T and an invariant measure μ_M. A stationary process ξ_t with continuous time can be constructed on the space $\Omega = M \times [0, \tau]$ (where τ is a constant) as a flow under a function[26]: The evolution of an initial point $\xi \equiv (\varphi, s) \in \Omega$ starts by moving with fixed φ and speed one from (φ, s) to (φ, τ); then it jumps to $(T\varphi, 0)$ and the cycle is restarted. This evolution has an invariant measure $d\mu = d\mu_M \cdot ds/\tau$. We now let $F(\xi_t)$ depend only on φ_t, i.e., we fix an initial time $t_0 \in [0, \tau]$, an initial value $\varphi \in M$, and an interval τ, and define a piecewise constant force

$$F(t) = f(T^k \varphi) \qquad \text{for} \quad t_0 + k\tau \leqslant t < t_0 + (k+1)\tau, \quad k = 0, 1, 2,... \tag{2.11}$$

or

$$F(t) \equiv \sum_{k=0}^{\infty} \chi_{[k\tau, (k+1)\tau)}(t - t_0)\, f(T^k \varphi) \tag{2.12}$$

where $\chi_{[t_1, t_2)}$ is the characteristic function of the interval $[t_1, t_2)$ and $f \in L_2(M, d\mu_M)$ satisfying $\int_M f(\varphi)\, d\mu_M = 0$.

In our two examples, M is taken to be the circle S^1, and the maps

(A) T: $\varphi \mapsto (\varphi + \alpha) \bmod 1, \quad \alpha \in [0, 1]$ irrational (2.13)

(B) T: $\varphi \mapsto (2\varphi) \bmod 1$ (2.14)

In both cases the stationary measure is $d\mu = d\varphi \cdot dt_0/\tau$. The dynamical system (2.13) is ergodic but not mixing (i.e., only weakly random), whereas (2.14) is a Bernoulli system (i.e., strongly chaotic). A similar model in which the values of the force are independent random variables has been studied in ref. 16.

To obtain the asymptotic behavior of the energy, it is enough to look at a discrete sequence of times $t_n = t_0 + n\tau$. Setting $z(n) \equiv z(t_n)$, Eq. (2.10) yields

$$\langle |z(n)|^2 \rangle_\mu = \frac{2}{\omega_0^4}(1 - \cos\omega_0\tau) \sum_{h=-n}^{n} (n - |h|)\, C_M(h) \cos(\omega_0 \tau h) \qquad (2.15)$$

where

$$C_M(h) = \int_M f(\varphi) f(T^h \varphi)\, d\mu_M(\varphi) \qquad (2.16)$$

is the correlation function of the discrete-time dynamical system. We notice that τ multiplies ω_0. Thus, the resonance conditions can be tuned up by changing the parameter τ of the force. In particular, if $\omega_0\tau$ is an integer multiple of 2π, the energy is always bounded, since $\langle |z|^2 \rangle_\mu$ vanishes.

3. DECAYING CORRELATIONS

The power spectrum (or spectral density) of the force by the Fourier transform is defined as

$$\tilde{C}(\omega) = \int_{-\infty}^{\infty} ds\, C(s) \cos \omega s \qquad (3.1)$$

Proposition 3.1. If the correlation satisfies

$$\int_0^{\infty} ds\, s|C(s)| < \infty \qquad (3.2)$$

and $\tilde{C}(\omega_0) \neq 0$, then

$$\langle E(t) \rangle_\mu \sim Dt \qquad \text{with} \qquad D = \frac{1}{\omega_0} \tilde{C}(\omega_0) \qquad (3.3)$$

Proof. It is immediate from Eq. (2.10):

$$\langle E(t) \rangle_\mu = \text{bounded fct.} - \int_{-t}^{t} ds\, |s| \frac{C(s)}{\omega_0} \cos \omega_0 s + t \frac{1}{\omega_0} \int_{-t}^{t} ds\, C(s) \cos \omega_0 s \qquad (3.4)$$

The first integral is bounded by the hypothesis (3.1) and

$$\lim_{t \to \infty} \frac{1}{t} \langle E(t) \rangle_{\mu} = \frac{1}{\omega_0} \tilde{C}(\omega_0) \tag{3.5}$$

For models with piecewise constant force (2.11), Proposition 3.1 becomes the following.

Proposition 3.2. If the dynamical system $(M, T, d\mu_M)$ and the function $f \in L_2(M, \mathbb{R}, d\mu_M)$ are such that the correlation function C_M of (2.16) satisfies

$$\sum_{h=0}^{\infty} h |C_M(h)| < \infty \tag{3.6}$$

and the power spectrum

$$\tilde{C}_M(\omega) \doteq \sum_{h=-\infty}^{\infty} C_M(h) \cos(h\omega) \tag{3.7}$$

is such that $\tilde{C}_M(\omega_0 \tau) \neq 0$, then (provided that $\omega_0 \tau$ is not an integer multiple of 2π) the energy grows linearly with time,

$$\langle E(t_n) \rangle_{\mu} \sim Dn \tag{3.8}$$

with

$$D = \frac{2}{\omega_0^4} (1 - \cos \omega_0 \tau) \tilde{C}_M(\omega_0 \tau) \tag{3.9}$$

Remark. It would of course be interesting to know when $\tilde{C}(\omega)$ is zero. We can say something about the zeros of $\tilde{C}(\omega)$ for some special cases:

(1) If $C(t)$ decays exponentially, then $\tilde{C}(\omega)$ is analytic and thus has at most a finite number of zeros in any finite interval.

(2) For certain piecewise constant forces, discussed in the next paragraph, which originate from a K-system, the set of zeros of the power spectrum is of measure zero.[27]

Example. We consider

$$T: \quad \varphi \to (2\varphi) \bmod 1 \tag{3.10}$$

and

$$f(\varphi) = \varphi - \tfrac{1}{2} \tag{3.11}$$

The correlation function is given by

$$C_M(h) = \tfrac{1}{12} 2^{-|h|} \tag{3.12}$$

and the power spectrum is

$$\tilde{C}_M(\omega_0 \tau) = \tfrac{1}{4}[5 - 4\cos(\omega_0 \tau)]^{-1} \tag{3.13}$$

The energy thus grows linearly with time. Figure 1 shows a typical time evolution $E(t_n)$.

For this example we can make a much stronger statement in the form of a central limit theorem. From Eq. (2.5) we can write the time dependence of $z(n)$ as

$$\begin{aligned} z_1(n) &= \mathrm{Re}\left\{\frac{1}{\omega_0^2}(e^{i\omega_0\tau} - 1)e^{i\omega_0 t_0} \sum_{k=0}^{n-1} e^{i\omega_0 \tau k} T^k \varphi\right\} \\ z_2(n) &= \mathrm{Im}\left\{\frac{1}{\omega_0^2}(e^{i\omega_0\tau} - 1)e^{i\omega_0 t_0} \sum_{k=0}^{n-1} e^{i\omega_0 \tau k} T^k \varphi\right\} \end{aligned} \tag{3.14}$$

The dispersion is asymptotically the same for the two components:

$$\begin{aligned} \langle z_i^2(n)\rangle_{\mu M} - \langle z_i(n)\rangle_{\mu M}^2 &= \frac{n(1-\cos\omega_0\tau)}{4(5-4\cos\omega_0\tau)} + o(n) \\ &\equiv \sigma_n^2 + o(n) \end{aligned} \tag{3.15}$$

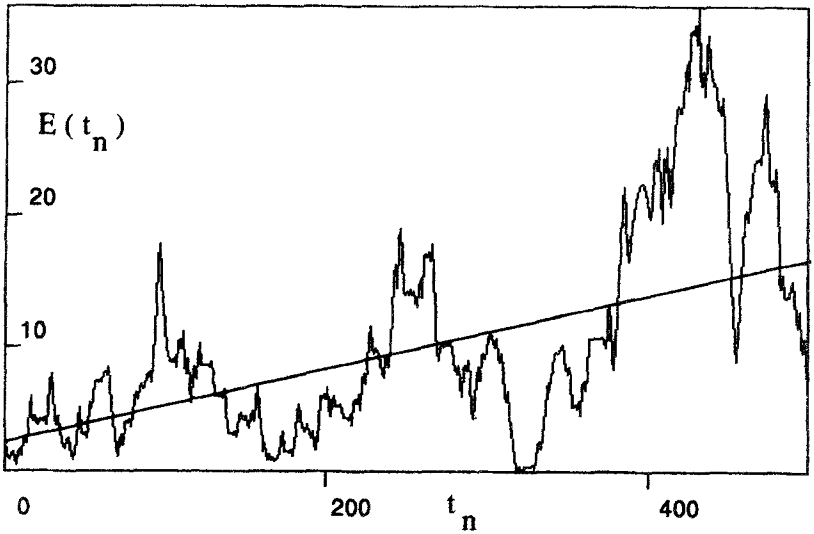

Fig. 1. Typical energy evolution $E(t_n)$ for the example (3.10). The straight line is the average $\langle E(t_n)\rangle_\mu$.

Theorem 3.3. The random variables $z_i(n)/\sigma_n$, $i = 1, 2$, converge in distribution to a standard Gaussian variable.

The proof is given in Appendix A.

4. NONDECAYING CORRELATIONS

In this section we study a family of quasiperiodic forces with two incommensurate frequencies which have a dense Fourier spectrum. $F(t)$ is a piecewise constant force described by

$$F(t) = f(T^k \varphi) \qquad \text{for} \quad k\tau + t_0 \leqslant t < (k+1)\tau + t_0 \tag{4.1}$$

with

$$\begin{aligned} T\varphi &= (\varphi + \alpha) \bmod 1, \qquad \alpha \in (0, 1), \text{ irrational} \\ f &\in L_2(S^1, \mathbb{R}, d\varphi), \qquad \int_0^1 d\varphi\, f(\varphi) = 0 \end{aligned} \tag{4.2}$$

The Fourier transform of $F(t)$ is equal to

$$\tilde{F}(\omega) = \text{const.} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_m e^{2\pi i \varphi m} \cdot \delta(\alpha m - l - \omega\tau) \tag{4.3}$$

where

$$a_m = \int_0^1 d\varphi\, f(\varphi) e^{-2\pi i m \varphi} \tag{4.4}$$

For this system

$$C_M(h) = \sum_{k=-\infty}^{\infty} |a_k|^2 e^{2\pi i h k \alpha} \tag{4.5}$$

This follows immediately from the Fourier representation (4.4):

$$f(T^h \varphi) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi i(\alpha h + \varphi)k} \tag{4.6}$$

Insertion into (2.15) yields the expression

$$\langle |z(n)|^2 \rangle_\mu = c \sum_{k=-\infty}^{\infty} |a_k|^2 \frac{\sin^2[n\pi(k\alpha - \hat{\tau})]}{\sin^2[\pi(k\alpha - \hat{\tau})]} \tag{4.7}$$

where $\hat{\tau} \times \omega_0 \tau / 2\pi$ and $c = 2(1 - \cos \omega_0 \tau)/\omega_0^4$. This expression shows that if $\hat{\tau} = (l\alpha) \bmod 1$, $l \in \mathbb{Z}$, the term $k = l$ of the sum produces a resonance and thus the energy will grow as n^2. If $\hat{\tau} \neq (l\alpha) \bmod 1$, the denominators are never zero, but infinitely many of them are arbitrarily close to zero. We have thus a "small-denominator" problem. The following two theorems show that the asymptotic behavior in n depends strongly on the properties of a_k for large k, or equivalently on the differentiability properties of the function f.

Theorem 4.1. Let $f(\varphi) = \varphi - 1/2$; then for all $\alpha, \hat{\tau} \in [0, 1)$ (α irrational) the energy is unbounded.

Theorem 4.2. If the function $f \in L_2(S^1, d\varphi)$ is such that its Fourier coefficients satisfy $\sum |a_k| < \infty$, then the energy of model (4.2) is bounded for all irrational $\alpha \in (0, 1)$ and almost all $\hat{\tau} \in [0, 1)$.

The essential difference is that in Theorem 4.1, f has a discontinuity, and thus the Fourier coefficients decay only as $1/k$.

Proof of Theorem 4.1. We remark that for fixed n the sum (4.7) is absolutely convergent, since $\sum |a_k|^2 < \infty$. For the present choice of f, the Fourier coefficients are

$$|a_k|^2 = \begin{cases} 0, & k = 0 \\ 1/(4\pi^2 k^2), & k \neq 0 \end{cases} \tag{4.8}$$

The idea of the proof is to find a sequence of times n_i, $i = 1, 2, \ldots$, such that

$$\lim_{i \to \infty} \langle |z(n_i)|^2 \rangle_\mu = \infty \tag{4.9}$$

To estimate the sum (4.7), we will use the following lemma of Tchebichev[28,29]:

Lemma 4.3. For all $\alpha, \hat{\tau} \in \mathbb{R}$, α irrational, there exists a sequence of positive integers k_ν, $\nu = 1, 2, \ldots$, such that

$$(k_\nu \alpha - \hat{\tau}) \bmod 1 < \frac{3}{k_\nu} \tag{4.10}$$

Using this subsequence $\{k_\nu\}$ and the fact that all terms in the sum (4.7) are positive, we have

$$\langle |z(n)|^2 \rangle_\mu \geq \frac{c}{4\pi^2} \sum_{\nu=1}^{\infty} \frac{\sin^2[n\pi(k_\nu \alpha - \hat{\tau})]}{k_\nu^2 \sin^2[\pi(k_\nu \alpha - \hat{\tau})]} \tag{4.11}$$

The denominators can be estimated using (4.10):

$$4\pi^2 k_\nu^2 \sin^2[\pi(k_\nu \alpha - \hat{\tau})] \leq 4\pi^4 k_\nu^2 [(k_\nu \alpha - \hat{\tau}) \bmod 1]^2 < c_1 = 36\pi^4 \tag{4.12}$$

Thus

$$\langle |z(n)|^2 \rangle_\mu > c_1 \sum_{\nu=1}^{\infty} \sin^2[n\pi(k_\nu \alpha - \hat{\tau})] \tag{4.13}$$

We will show now that for any integer m there is an n_m such that

$$\langle |z(n_m)|^2 \rangle_\mu > \tfrac{1}{2} c_1 m \tag{4.14}$$

Since the sequence $\tilde{\theta}_\nu \equiv (k_\nu \alpha - \hat{\tau}) \bmod 1$ tends to zero, we can extract a subsequence θ_l, $l = 1, 2,...,$ such that

$$\theta_1 < \tfrac{1}{4} \tag{4.15}$$

$$\theta_{l+1} < \tfrac{1}{4}\theta_l \tag{4.16}$$

and we have, fixing an arbitrary m,

$$\langle |z(n)|^2 \rangle_\mu > c_1 \left(\sum_{l=1}^{m} \sin^2[n\pi\theta_l] + \sum_{l=m+1}^{\infty} \sin^2[n\pi\theta_l] \right) \tag{4.17}$$

We now use the following lemma.

Lemma 4.4. For each m there is a value n_m such that each of the first m terms satisfies

$$\tfrac{1}{4} < (n_m \theta_l) \bmod 1 < \tfrac{3}{4} \qquad (l = 1,..., m) \tag{4.18}$$

This lemma implies

$$\sum_{l=1}^{m} \sin^2[\pi n_m \theta_l] > \tfrac{1}{2} m \tag{4.19}$$

and therefore by (4.17) the energy is unbounded.

Proof of Lemma 4.4. We define $l_1,..., l_m$ recursively as follows: l_s is the smallest integer such that

$$l_s \geq l_{s-1} \qquad \text{and} \qquad \tfrac{1}{4} < (l_s \theta_{m-s+1}) \bmod 1 < \tfrac{3}{4} \tag{4.20}$$

and

$$\tfrac{1}{4} < l_1 \theta_m < \tfrac{3}{4} \tag{4.21}$$

We will show that the choice $n_m = l_m$ satisfies the required property (5.18).

We define the quantity R_s by

$$l_m \theta_{m+1-s} = l_s \theta_{m+1-s} + R_s, \qquad s = 1,..., m \tag{4.22}$$

It is enough to show that

$$0 \leqslant R_s < \tfrac{1}{4} \tag{4.23}$$

since by definition

$$\tfrac{1}{4} < (l_s \theta_{m+1-s}) \bmod 1 < \tfrac{1}{4} + \theta_{m+1-s} < \tfrac{1}{4} + (\tfrac{1}{4})^{m+1-s}$$

We use the following relations:

$$R_s = (l_m - l_s)\theta_{m+1-s} = \sum_{j=s+1}^{m} (l_j - l_{j-1})\,\theta_{m+1-s} \tag{4.24}$$

$$(l_j - l_{j-1})\theta_{m+1-s} \leqslant (l_j - l_{j-1})\,\theta_{m+1-j}\frac{1}{4^{j-s}}; \qquad \text{for} \quad j > s \tag{4.25}$$

$$(l_j - l_{j-1})\theta_{m+1-j} < \tfrac{1}{2} \tag{4.26}$$

Relation (4.25) is a consequence of (4.16). Relation (4.26) states the fact that, if $l_{j-1}\theta_{m+1-j}$ is not in the desired interval $[r_j + 1/4,\ r_j + 3/4]$, the distance to it is at most 1/2 and thus will be reached in less than $(2\theta_{m+1-j})^{-1}$ supplementary steps.

We obtain thus the result

$$0 \leqslant R_s \leqslant \frac{1}{2} \sum_{j=s+1}^{m} \frac{1}{4^{j-s}} < \frac{1}{4} \tag{4.27}$$

which completes the proof of the lemma.

Proof of Theorem 4.2. We start by writing

$$\begin{aligned}\langle |z_n|^2 \rangle_\mu &= c \sum_{k=-\infty}^{\infty} |a_k|^2 \frac{\sin^2[n\pi(k\alpha - \hat{\tau})]}{\sin^2[\pi(k\alpha - \hat{\tau})]} \\ &\leqslant c \sum_{k=-\infty}^{\infty} |a_k|^2 \frac{1}{\sin^2[\pi(k\alpha - \hat{\tau})]}\end{aligned} \tag{4.28}$$

We then separate the sum into three parts:

$$\sum_{k=-\infty}^{\infty} = \sum_{k \in I_1} + \sum_{k \in I_2} + \sum_{k \in I_3} \tag{4.29}$$

where

$$\begin{aligned}I_1 &= \{k \mid -\tfrac{1}{2} \leqslant (k\alpha) \bmod 1 - \hat{\tau} \leqslant \tfrac{1}{2}\} \\ I_2 &= \{k \mid -1 \leqslant (k\alpha) \bmod 1 - \hat{\tau} \leqslant -\tfrac{1}{2}\} \\ I_3 &= \{k \mid \tfrac{1}{2} < (k\alpha) \bmod 1 - \hat{\tau} \leqslant 1\}\end{aligned} \tag{4.30}$$

Adapting the inequality

$$|\sin\theta| \geqslant \frac{2}{\pi}|\theta| \qquad \text{if} \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{4.31}$$

to these three sets, we obtain:

(i) If $k \in I_1$:

$$|\sin[\pi(k\alpha - \hat{\tau})| = |\sin\{\pi[(k\alpha) \bmod 1 - \hat{\tau}]\}| > 2|(k\alpha) \bmod 1 - \hat{\tau}| \tag{4.32}$$

(ii) If $k \in I_2$:

$$|\sin[\pi(k\alpha - \hat{\tau})]| = |-\sin\{\pi[(k\alpha) \bmod 1 - \hat{\tau} + 1]\}| > 2|(k\alpha) \bmod 1 - \hat{\tau} + 1| \tag{4.33}$$

(iii) If $k \in I_3$:

$$|\sin[\pi(k\alpha - \hat{\tau})]| = |-\sin\{\pi[(k\alpha) \bmod 1 - \hat{\tau} - 1]\}| > 2|(k\alpha) \bmod 1 - \hat{\tau} - 1| \tag{4.34}$$

Applying these estimates, we can write

$$\langle |z_n|^2 \rangle_\mu \leqslant \frac{c}{4} \sum_{k=-\infty}^{\infty} \frac{|a_k|^2}{|s_k - \hat{\tau}|^2} \tag{4.35}$$

with

$$s_k = \begin{cases} (k\alpha) \bmod 1 & \text{if} \quad k \in I_1 \\ (k\alpha) \bmod 1 + 1 & \text{if} \quad k \in I_2 \\ (k\alpha) \bmod 1 - 1 & \text{if} \quad k \in I_3 \end{cases} \tag{4.36}$$

This expression is bounded as a consequence of the following lemma due to Howland, which completes the proof.

Lemma.[30] Let $\{s_k\}$, $-\infty < k < \infty$, be an arbitrary sequence on some finite interval $J \subset \mathbb{R}$. If $\sum_{k=-\infty}^{\infty} |a_k| < \infty$, then

$$\sum_{k=-\infty}^{\infty} \frac{|a_k|^2}{|s_k - \hat{\tau}|^2} < \infty$$

for almost all $\hat{\tau} \in J$.

5. QUANTUM MECHANICAL SYSTEMS

As already mentioned, there is a close relation between the classical and quantum evolution of harmonic oscillators. The quantum evolution can be entirely deduced from the classical motion in phase space.[4,16] The results for the energy growth are in fact identical for the two systems. This can be readily seen by stating the quantum problem in terms of the Wigner function. The time evolution of states $\psi \in L_2(\mathbb{R}, dx)$ is determined by the Schrödinger equation

$$i \frac{\partial}{\partial t} \psi_t = H(t) \psi_t \equiv [H_0 - xF(t)] \psi_t \tag{5.1}$$

(we set $\hbar = 1$). The state ψ_t at time t can be expressed in terms of the Wigner distribution P_w defined as[31]

$$P_w(q, p, t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} dy\, \psi_t^*(q + y)\, \psi_t(q - y) \exp[2i\pi yp] \tag{5.2}$$

P_w is normalized, but not necessarily positive. For the forced harmonic oscillator the time evolution of P_w is determined by the equation[31]

$$\frac{\partial}{\partial t} P_w = -p \frac{\partial}{\partial q} P_w + (\omega^2 q - F) \frac{\partial}{\partial p} P_w \tag{5.3}$$

which coincides with the classical Liouville equation. The solution can thus be expressed as

$$P_w(q, p, t) = P_w(q_{-t}, p_{-t}, 0) \equiv P_w(\bar{q}, \bar{p}, 0) \tag{5.4}$$

where $\bar{q}$, $\bar{p}$ are the initial conditions that would evolve into q, p at time t. We denote this by

$$q = q_t(\bar{q}, \bar{p}), \qquad p = p_t(\bar{q}, \bar{p}) \tag{5.5}$$

We consider initial states ψ_0 with finite energy expectation $\langle \psi_0, H\psi_0 \rangle < \infty$. Its time evolution can then be expressed as

$$\begin{aligned} \langle \psi_t, H\psi_t \rangle &= \int dq\, dp \left(\frac{1}{2} p^2 + \frac{\omega_0^2}{2} q^2 \right) P_w(q, p, t) \\ &= \int d\bar{q}\, d\bar{p} \left[\frac{1}{2} p_t^2(\bar{q}, \bar{p}) + \frac{\omega_0^2}{2} q_t^2(\bar{q}, \bar{p}) \right] P_w(\bar{q}, \bar{p}, 0) \end{aligned} \tag{5.6}$$

When averaged over the forcing process, we obtain

$$
\begin{aligned}
E^Q(\psi; t) &\equiv \langle\langle \psi_t, H\psi_t \rangle\rangle_\mu \\
&= \int d\bar{q}\, d\bar{p} \left\langle \frac{1}{2} p_t^2(\bar{q}, \bar{p}) + \frac{\omega_0^2}{2} q_t^2(\bar{q}, \bar{p}) \right\rangle_\mu P_w(\bar{q}, \bar{p}, 0) \\
&= \int d\bar{q}\, d\bar{p}\, E^c(\bar{q}, \bar{p}; t)\, P_w(\bar{q}, \bar{p}, 0)
\end{aligned}
\tag{5.7}
$$

and inserting (2.9)

$$E^Q(\psi_0; t) = E^Q(\psi_0; 0) + \langle |z(t)|^2 \rangle_\mu \tag{5.8}$$

Thus, the averaged energy in the quantum model has the same growth properties as the classical one, determined by $\langle |z(t)|^2 \rangle_\mu$.

We now turn to the analysis of the spectrum of the Floquet operator in the case of periodic time dependence. We verify explicitly that the spectrum is pure point in the absence of a resonance and absolutely continuous in the presence of one. The Floquet operator $U(T, 0)$ is defined in $\mathscr{H} = L_2(\mathbb{R}, dx)$ by the propagator of (5.1) taken over one period T. Its spectrum is directly related to the spectrum of the "quasi-energy" operator

$$K = -i \frac{\partial}{\partial t} + H(t) \tag{5.9}$$

in the space $L_2(S_T^1, dt/T) \otimes L_2(\mathbb{R}, dx)$, where S_T^1 is a circle of length T. The eigenvalues and eigenvectors of the two operators are related by

$$
\begin{aligned}
K\psi &= \lambda\psi, \qquad \psi(x, t) \in L_2(\mathbb{R}, dx) \otimes L_2\left(S_T^1, \frac{dt}{T}\right) \\
U(T, 0)\phi &= e^{-i\lambda T}\phi; \qquad \phi(x) \in L_2(\mathbb{R}, dx) \\
\psi(x, t) &= e^{i\lambda t} U(t, 0)\, \phi(x)
\end{aligned}
\tag{5.10}
$$

The quasi-energy operator was introduced in refs. 32 and 10 and has since proven to be a useful tool to treat time-dependent problems. In refs. 11 and 13 it has been shown under some conditions that the spectrum of the Floquet operator characterizes the dynamics of the states ψ (RAGE theorem[(33)]):

(A) ψ belongs to the subspace of point spectrum of $U(T, 0)$ if and only if $\forall \varepsilon > 0$, $\exists R > 0$ such that

$$\inf_t \int_{|x| < R} dx\, |U(t, s)\, \psi(x)|^2 \geqslant (1 - \varepsilon)\, \|\psi\|^2 \tag{5.11}$$

(B) ψ belongs to the subspace of continuous spectrum of $U(T, 0)$ if and only if $\forall R > 0$

$$\lim_{\tau \to \pm\infty} \frac{1}{\tau} \int_0^\tau dt \int_{|x| < R} dx \, |U(t, s) \, \psi(x)|^2 = 0 \tag{5.12}$$

Furthermore, we will show that the energy remains bounded if the spectrum is pure point, and it grows unbounded if the spectrum is continuous. To make this statement more precise, we introduce the following subspaces: Let $\mathscr{H}_{\text{pp}}$ and $\mathscr{H}_{\text{c}}$ be the pure point and continuous spectrum subspaces, respectively, corresponding to the Floquet operator $U(T, 0)$. The subspace $\mathscr{H}_{\text{be}}$ of state trajectories with bounded energy is defined as

$$\mathscr{H}_{\text{be}} = \{\psi \in \mathscr{H} \mid \lim_{E \to \infty} \sup_{t \geq 0} \|F(H_0 > E) \, U(t, 0)\psi\| = 0\} \tag{5.13}$$

where $F(H_0 > E)$ is the spectral projection on the eigenspace of H_0 corresponding to energies larger than E. The subspace of states with precompact trajectories is defined as

$$\mathscr{H}_{\text{pc}} = \{\psi \in \mathscr{H} \mid \text{closure of } \{U(t, 0)\psi, t \geq 0\} \text{ is compact}\} \tag{5.14}$$

i.e., the trajectories in $\mathscr{H}_{\text{pc}}$ evolve in a space of finite dimension except for a small correction. More precisely, given $\psi \in \mathscr{H}_{\text{pc}}$, there is for any $\varepsilon > 0$ a decomposition

$$\psi(t) = \psi_f(y) + v(t) \tag{5.15}$$

such that for all $t \geq 0$, $\psi_f(t)$ is a finite-dimensional subspace and $\|v(t)\| < \varepsilon$.

Theorem 5.1. Consider the time-dependent Hamiltonian

$$H(t) = H_0 + \hat{V}(x, t); \qquad \hat{V}(x, t + T) = \hat{V}(x, t) \tag{5.16}$$

such that the spectrum of the unperturbed H_0 is discrete and bounded from below (e.g., having a confining potential, or defined on a compact manifold) and such that the propagator $U(t, s)$ exists as a strongly continuous family of unitary operators. Then

$$\mathscr{H}_{\text{pp}} = \mathscr{H}_{\text{be}}; \qquad \mathscr{H}_{\text{c}} = \mathscr{H}_{\text{be}}^{\perp} \tag{5.17}$$

We remark that $\psi \in \mathscr{H}_{\text{be}}$ does not imply that the expectation value $\langle \psi(t), H_0 \psi(t) \rangle$ stays bounded: although the components of higher energies decrease to zero, the decay can be slow enough to give a diverging expectation.

For the proof we use the following two theorems due to V. Enss and K. Veselic:

Theorem 5.2.[13] Let C be any compact operator and $\psi \in \mathscr{H}_c$; then there is a nonnegative function $f(\tau)$ such that

$$\frac{1}{\tau}\int_0^\tau dt\, \|CU(t,0)\psi\| \leqslant f(\tau)\,\|\psi\| \tag{5.18}$$

with

$$\lim_{|\tau|\to\infty} f(\tau)=0 \tag{5.19}$$

Theorem 5.3.[13] $\mathscr{H}_{pp}=\mathscr{H}_{pc}$.

Proof of Theorem 5.1. (i) We start by proving $\mathscr{H}_{be}\subset\mathscr{H}_{pp}$: Consider a $\psi\in\mathscr{H}_c$. Applying Theorem 5.2 with $C=F(H_0\leqslant E)$ to the first term in the identity

$$1=\|\psi\|=\frac{1}{\tau}\int_0^\tau dt\, \|F(H_0\leqslant E)\,U(t,0)\psi\| + \|F(H_0>E)\,U(t,0)\psi\| \tag{5.20}$$

we obtain

$$\frac{1}{\tau}\int_0^\tau dt\, \|F(H_0>E)\,U(t,0)\psi\| \geqslant [1-f(\tau)] \tag{5.21}$$

This implies that

$$\sup_{t\in[0,\tau]} \|F(H_0>E)\,U(t,0)\psi\| \geqslant [1-f(\tau)] \tag{5.22}$$

and

$$\sup_{t\in[0,\infty)} \|F(H_0>E)\,U(t,0)\psi\| = 1 \qquad (\text{for any } E) \tag{5.23}$$

Hence

$$\lim_{E\to\infty}\ \sup_{t\geqslant 0} \|F(H_0>E)\,U(t,0)\psi\| = 1 \tag{5.24}$$

i.e., states of the continuous spectrum subspace have trajectories with unbounded energy.

(ii) The inclusion in the other direction $\mathscr{H}_{pp}\subset\mathscr{H}_{be}$ follows from Theorem 5.3 together with $\mathscr{H}_{pc}\subset\mathscr{H}_{be}$, which can be shown as follows:

Given $\psi \in \mathscr{H}_{pc}$ and $\varepsilon > 0$, the precompactness implies that there is a decomposition

$$\psi(t) = \psi_f(t) + v(t) \tag{5.25}$$

such that for all $t \geqslant 0$, $\psi_f(t)$ is in a finite-dimensional subspace and $\|v(t)\| < \varepsilon$. Then

$$\sup_{t\geqslant 0} \|F(H_0 > E)\,\psi(t)\| \leqslant \sup_{t\geqslant 0} \|F(H_0 > E)\,\psi_f(t)\| + \sup_{t\geqslant 0} \|F(H_0 > E)\,v(t)\| \tag{5.26}$$

The first term is finite dimensional and thus tends to zero when $E \to \infty$ and the second one is bounded by ε, and thus

$$\lim_{E\to\infty} \sup_{t\geqslant 0} \|F(H_0 > E)\,\psi(t)\| < \varepsilon, \qquad \forall \varepsilon > 0 \tag{5.27}$$

which completes the proof.

For periodic Hamiltonians that are quadratic polynomials in p and q, Hagedorn *et al.*[16] have shown that the spectrum is either pure point or transient absolutely continuous. They also classified the models in relation with the classical motion. Their method applied to the present model (5.1) allows one to calculate the spectrum and eigenfunctions explicitly: The propagator can be expressed as[13]

$$U(t, 0) = e^{-i\delta(t)} e^{i\sigma_2(t)x} e^{-i\sigma_1(t)\hat{p}} e^{-iH_0 t} \tag{5.28}$$

where $\hat{p} = -i\,\partial/\partial x$ and

$$\sigma_1(t) = \frac{1}{\omega_0} \int_0^t ds\, F(s) \sin \omega_0(t-s)$$

$$\sigma_2(t) = \int_0^t ds\, F(s) \cos \omega_0(t-s) \tag{5.29}$$

$$\delta(t) = \frac{1}{2} \int_0^t ds\, [\sigma_2^2(s) - \omega_0^2 \sigma_1^2(s)]$$

(This is easily verified by applying it on a basis of coherent states of H_0.)

We assume that $F(t+T) = F(t)$, which defines the frequency $\omega = 2\pi/T$. We have to distinguish two cases.

I. *Resonance*: $\omega_0 = k\omega$, $k \in \mathbb{Z}$. In this case the operator $e^{-iH_0 T}$ reduces to multiplication with a constant. This is easily seen by applying to to the basis $\{\varphi_n\}$ of eigenfunctions of H_0:

$$e^{-iH_0 T} \varphi_n = e^{-i(n+1/2)\omega_0 T} \varphi_n = e^{-i\pi k} \varphi_n \tag{5.30}$$

Thus

$$U(T,0)=e^{-i\delta(T)-i\pi k}e^{+i\sigma_2(T)x}e^{-i\sigma_1(T)\hat{p}} \tag{5.31}$$

which has transient absolutely continuous spectrum.[16]

II. *Nonresonance*: $\omega_0 \neq k\omega$, $k \in \mathbb{Z}$.

We will use the following result.

Lemma. If $\omega_0 t \neq k2\pi$, $k \in \mathbb{Z}$, then for any $\alpha, \beta \in \mathbb{R}$ there are $\gamma, \delta, \theta \in \mathbb{R}$ such that

$$e^{i\beta x}e^{-i\alpha\hat{p}}e^{-iH_0 t}=e^{i\theta}e^{i(\delta x-\gamma\hat{p})}e^{-iH_0 t}e^{-i(\delta x-\gamma\hat{p})} \tag{5.32}$$

with

$$\begin{pmatrix}\gamma\\ \delta/\omega_0\end{pmatrix}=(1-R_t)^{-1}\begin{pmatrix}\alpha\\ \beta/\omega_0\end{pmatrix} \tag{5.33}$$

where R_t is the rotation matrix (2.4).

Proof. It follows immediately, from applying (5.32) to a basis of coherent states of the form

$$\varphi=e^{ixp_0}e^{-\omega_0(x-q_0)^2/2} \tag{5.34}$$

and using the fact that

$$e^{-iH_0 t}\varphi=e^{i\delta'_t}e^{ixp_t}e^{-\omega_0(x-q_t)^2/2} \tag{5.35}$$

where

$$\begin{pmatrix}q_t\\ p_t/\omega_0\end{pmatrix}=R_t\begin{pmatrix}q_0\\ p_0/\omega_0\end{pmatrix} \tag{5.36}$$

and δ'_t is a phase. We remark that the condition of nonresonance is used only for the existence of the inverse of $(1-R_t)$.

The Floquet operator $U(T,0)$ is thus unitarily equivalent (up to a phase) to $e^{-iH_0 T}$ and has pure point spectrum with eigenvalues

$$e^{-i\lambda_n T}=e^{-i\omega_0(n+1/2)T-i\theta T}, \qquad n=0,1,2,... \tag{5.37}$$

where θ is a constant.

The eigenfunctions ϕ_n can be expressed in terms of the eigenfunctions φ_n of H_0:

$$\phi_n(x)=e^{i(\delta x-\gamma\hat{p})}\varphi_n(x)=ce^{i\delta x}\varphi_n(x-\gamma) \tag{5.38}$$

where

$$\begin{pmatrix} \delta \\ \gamma/\omega_0 \end{pmatrix} = (1 - R_T)^{-1} \begin{pmatrix} \sigma_1(T) \\ \sigma_2(T)/\omega_0 \end{pmatrix}$$

The eigenvalues λ and eigenfunctions $\psi_{n,m}$ of the generator $K = -i\,\partial/\partial t + H(t)$ can then be expressed as

$$\begin{aligned} \lambda_{n,m} &= \omega m + \omega_0(n + \tfrac{1}{2}) + \theta; \qquad n = 0, 1, 2,...; \quad m \in \mathbb{Z} \\ \psi_{n,m}(x, t) &= e^{im\omega t}\psi_n(x, t) \end{aligned} \tag{5.39}$$

where

$$\psi_n(x, t) \equiv e^{i\lambda_n t}\, U(t, 0)\, \phi_n(x) \tag{5.40}$$

This example illustrates the following property, which is expected to be quite general:

(i) If ω and ω_0 are commensurate ($\omega_0 = (k/h)\omega$, $h \neq 1$, $k, h \in \mathbb{Z}$, relatively prime), then there is a finite number of eigenvalues λ and they have infinite degeneracy.

(ii) If ω and ω_0 are not commensurate, the point spectrum is non-degenerate and dense.

This is the origin of small-denominator problems in the perturbation analysis of such operators.[17,20]

APPENDIX

Proof of Theorem 3.3. We show that the characteristic function satisfies

$$\phi_n(\lambda) \doteq \langle e^{i\lambda z_i(n)/\sigma_n} \rangle \xrightarrow[n \to \infty]{} z^{-\lambda^2/2}, \qquad \text{for} \quad i = 1, 2 \tag{A.1}$$

We have a sum of weakly dependent random variables, but multiplied with nonstationary (i.e., k-dependent) coefficients. We cannot therefore apply the standard theorems. But by a suitable change of variables we can transform the problem into a sum independent random variables with non-stationary coefficients, which can be easily estimated. We represent the points $\varphi \in [0, 1]$ by their binary expansion

$$\varphi = \sum_{p=1}^{\infty} \frac{a_p}{2^p}, \qquad a_p \in \{0, 1\} \tag{A.2}$$

The uniform measure $d\varphi$ is equivalent to the Bernoulli measure on the sequences $\{a_p\}$ (independent equal probability 1/2 for 0 and 1 at each position).

The map T acts as a shift and we can write

$$\begin{aligned} z_1(n) &= \mathrm{Re}\left\{\frac{1}{\omega_0^2}(e^{i\omega_0\tau}-1)e^{i\omega_0 t_0}\sum_{k=0}^{\infty} e^{i\omega_0\tau k}\sum_{p=1}^{\infty}\frac{a_{p+k}}{2^p}\right\} \\ z_2(n) &= \mathrm{Im}\left\{\frac{1}{\omega_0^2}(e^{i\omega_0\tau}-1)e^{i\omega_0 t_0}\sum_{k=0}^{\infty} e^{i\omega_0\tau k}\sum_{p=1}^{\infty}\frac{a_{p+k}}{2^p}\right\} \end{aligned} \tag{A.3}$$

Making the change of variables $l=p+k$, $p'=p$,

$$\sum_{k=0}^{n-1}\sum_{p=1}^{\infty} = \sum_{p'=1}^{\infty}\sum_{l=p'}^{p'+n-1} = \sum_{l=1}^{n-1}\sum_{p'=1}^{l} + \sum_{l=n}^{\infty}\sum_{p'=l-n+1}^{l} \tag{A.4}$$

we can perform the sums over p' and obtain

$$z_i(n) = \sum_{l=1}^{\infty} a_l g_{l,n}^{(i)}, \qquad i=1,2 \tag{A.5}$$

with

$$g_{l,n}^{(1)} = \mathrm{Re}\{h_{l,n}\}; \qquad g_{l,n}^{(2)} = \mathrm{Im}\{h_{l,n}\} \tag{A.6}$$

$$h_{l,n} = \begin{cases} \dfrac{(e^{i\omega_0\tau}-1)e^{i\omega_0 t_0}}{(2e^{i\omega_0\tau}-1)\omega_0^2}\left(e^{il\omega_0 t_0}-\dfrac{1}{2^l}\right) & \text{for } 1 \leqslant l \leqslant n-1 \quad \text{(A.7a)} \\ \dfrac{(e^{i\omega_0\tau}-1)e^{i\omega_0 t_0}}{(2e^{i\omega_0\tau}-1)\omega_0^2}\left(\dfrac{e^{i\omega_0\tau}}{2^{l-n}}-\dfrac{1}{2^l}\right) & \text{for } l \geqslant n \quad \text{(A.7b)} \end{cases}$$

We will only need the following properties of $g_{l,n}^{(i)}$:

(a) $g_{l,n}^{(i)}$ is bounded in l, n

(b) $$\sum_{l=n}^{\infty} |g_{l,n}^{(i)}| < c_0 \tag{A.8}$$

(c) $$\left|\sum_{l=1}^{n-1} g_{l,n}^{(i)}\right| < c_0' \tag{A.9}$$

(d) $$\lim_{n\to\infty}\frac{1}{4\sigma_n^2}\sum_{l=1}^{n-1}(g_{l,n}^{(i)})^2 = 1 \tag{A.10}$$

where c_0, c_0' are constants, independent of n. The property (A.8) follows from the fact that if $l \geqslant n$, then for large n

$$|g_{l,n}^{(i)}| \sim \text{const.}\cdot\frac{1}{2^{l-n}} \tag{A.11}$$

Property (A.9) follows immediately from (A.7a), and (A.10) is a consequence of

$$\sigma_n^2 \sim \langle |z^{(i)}|^2 \rangle = \frac{1}{4} \sum_{l=1}^{\infty} (g_{l,n}^{(i)})^2 \sim \frac{1}{4} \sum_{l=1}^{n-1} (g_{l,n}^{(i)})^2 \tag{A.12}$$

We can now evaluate the characteristic function

$$\begin{aligned}\phi_n(\lambda) &= \left\langle \exp\left(i \frac{\lambda}{\sigma_n} \sum_{l=1}^{\infty} a_l g_{l,n}^{(i)} \right) \right\rangle = \prod_{l=1}^{\infty} \left\langle \exp\left(i \frac{\lambda}{\sigma_n} a_l g_{l,n}^{(i)} \right) \right\rangle \\ &= \prod_{l=1}^{\infty} \left(\frac{1}{2} + \frac{1}{2} \exp \frac{i\lambda g_{l,n}^{(i)}}{\sigma_n} \right) \end{aligned} \tag{A.13}$$

We take the logarithm and separate the terms according to (A.7),

$$\ln \phi_n(\lambda) = \sum_{l=1}^{n-1} \ln[1 + \tfrac{1}{2}(e^{i\lambda g_{l,n}^{(i)}/\sigma_n} - 1)] + \sum_{l=n}^{\infty} \ln[1 + \tfrac{1}{2}(e^{i\lambda g_{l,n}^{(i)}/\sigma_n} - 1)] \tag{A.14}$$

We can easily estimate the second sum using the property (A.8) and the inequality $|\ln(1+y)| < c_1 |y|$; c_1 constant (for n large):

$$\begin{aligned} &\left| \sum_{l=n}^{\infty} \ln[1 + \tfrac{1}{2}(e^{i\lambda g_{l,n}^{(i)}/\sigma_n} - 1)] \right| \\ &\quad \leqslant \frac{c_1}{2} \sum_{l=n}^{\infty} |e^{i\lambda g_{l,n}^{(i)}/\sigma_n} - 1| \\ &\quad \leqslant \frac{c_1'}{2} \frac{\lambda}{\sigma_n} \sum_{l=n}^{\infty} |g_{l,n}^{(i)}| \leqslant c_2 \frac{\lambda}{\sqrt{n}} \xrightarrow[n \to \infty]{} 0 \end{aligned} \tag{A.15}$$

The first sum of (A.14) can be estimated as follows: we define

$$y_l \doteq \tfrac{1}{2}(e^{i\lambda g_{l,n}^{(i)}/\sigma_n} - 1) \tag{A.16}$$

and use the inequality

$$|\ln(1+y) - y + \tfrac{1}{2} y^2| < c_3 |y|^3 \tag{A.17}$$

to write

$$\sum_{l=1}^{n-1} \ln[1 + y_l] = \sum_{l=1}^{n-1} [y_l - \tfrac{1}{2} y_l^2] + R_n \tag{A.18}$$

with

$$|R_n| \leqslant c_3 \sum_{l=1}^{n-1} |y_l|^3 \tag{A.19}$$

Further, since g_l is bounded,

$$|y_l| \leqslant \frac{\lambda}{\sigma_n} |g_{l,n}^{(i)}| \leqslant c_4 \frac{\lambda}{\sqrt{n}} \tag{A.20}$$

and thus

$$|R_n| \leqslant c_5 \lambda^3 \frac{n}{n^{3/2}} \xrightarrow[n \to \infty]{} 0 \tag{A.21}$$

For the first term in (A.18) we use

$$y_l - \tfrac{1}{2} y_l^2 = i\lambda g_{l,n}^{(i)}/\sigma_n - \tfrac{1}{8}\lambda^2 (g_{l,n}^{(i)})^2 \sigma_n + o(n) \tag{A.22}$$

which together with properties (A.9) and (A.10) of $g_{l,n}^{(i)}$ gives

$$\lim_{n \to \infty} \sum_{l=1}^{n-1} [y_l - \tfrac{1}{2} y_l^2] = -\tfrac{1}{2}\lambda^2 \tag{A.23}$$

Thus, putting the estimates together, we get

$$\lim_{n \to \infty} \ln \phi_n(\lambda) = -\tfrac{1}{2}\lambda^2$$

which completes the proof of the central limit theorem. ∎

ACKNOWLEDGMENTS

We would like to thank G. Gallavotti, J. S. Howland, R. T. Bumby, M. Combescure, and C.-A. Pillet for very helpful discussions. H.R.J. was supported by grant 8220-025973 of the Swiss National Science Foundation. A.P. thanks the Italian CNR for supporting his visit to Rutgers University. The work of L.B., J.L.L., and A.P. was supported in part by AFOSR grant 87-0010.

REFERENCES

1. J. E. Bayfield and P. M. Koch, *Phys. Rev. Lett.* **33**:258 (1974).
2. J. G. Leopold and I. C. Percival, *Phys. Rev. Lett.* **41**:944 (1978).
3. N. B. Delone, B. P. Krainov, and D. L. Shepelyansky, *Sov. Phys. Usp.* **26**:551 (1983); G. Casati, B. V. Chirikov, and D. L. Shepelyansky, *Phys. Rev. Lett.* **53**:2525 (1984); K. van Leeuwen, G. V. Oppen, S. Renwick, J. Boowlin, P. Koch, R. Jensen, O. Rath, D. Richards, and G. Leopold, *Phys. Rev. Lett.* **55**:2231 (1985).
4. M. Combescure, *Ann. Inst. H. Poincaré* **44**:293 (1986); *Ann. Phys.* **173**:210 (1987).
5. M. Combescure, Crystallization of trapped ions—A quantum approach, Preprint LPTHE Orsay (1989).

6. R. Jensen, *Phys. Rev. Lett.* **49**:1365 (1982).
7. J. S. Howland, in *Mathematical Methods and Applications of Scattering Theory*, J. A. de Santo, ed. (Springer-Verlag, Berlin, 1980).
8. S. Washburn and R. A. Webb, *Adv. Phys.* **35**:375 (1986).
9. J. Bellissard, in *Stochastic Processes in Classical and Quantum Systems*, S. Albeverio, G. Casati, and D. Merlini, eds. (Springer-Verlag, Berlin, 1986).
10. J. S. Howland, *Math. Ann.* **207**:315 (1974); *Indiana Univ. Math. J.* **28**:471 (1979).
11. K. Yajima and H. Kitada, *Ann. Inst. H. Poincaré A* **39**:145 (1983).
12. K. Yajima, *Commun. Math. Phys.* **87**:331 (1982).
13. V. Enss and K. Veselic, *Ann. Inst. H. Poincaré A* **39**:159 (1983).
14. G. Casati and L. Molinari, *Prog. Theor. Phys. Suppl.* **98**:287 (1989).
15. S. Fishman, D. R. Grempel, and R. E. Prange, *Phys. Rev. Lett.* **49**:509 (1982).
16. G. A. Hagedorn, M. Loss, and J. Slawny, *J. Phys. A* **19**:521 (1986).
17. J. Bellissard, in *Trends in the Eighties*, S. Albeverio and Ph. Blanchard, eds. (World Scientific, Singapore, 1985).
18. P. Seba, *Phys. Rev. A* **41**:2306 (1990).
19. J. S. Howland, *Ann. Inst. H. Poincaré* **49**:309, 325 (1989).
20. M. Combescure, *Ann. Inst. H. Poincaré* **47**:63 (1987); *Ann. Phys.* **185**:86 (1988).
21. D. L. Shepelyansky, *Physica* **8D**:208 (1983).
22. I. Guarneri, *Lett. Nuovo Cim.* **40**:171 (1984).
23. C.-A. Pillet, *Commun. Math. Phys.* **102**:237 (1985); **105**:259 (1986).
24. C.-A. Pillet, in *Stochastic Processes in Classical and Quantum Systems*, S. Albeverio, G. Casati, and D. Merlini, (Springer-Verlag, Berlin, 1986).
25. T. Dittrich and R. Graham, *Europhys. Lett.* 7:287 (1988).
26. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer-Verlag, Berlin, 1982).
27. I. A. Ibragimov and Y. V. Linnik, *Independent and Stationary Sequences of Random Variables* (Wolters-Noordhoff, Groningen, 1971).
28. A. Khintchin, *Continued Fractions* (Noordhoff, Groningen, 1963).
29. J. W. S. Cassels, *An Introduction to Diophantine Approximation* (Cambridge University Press, 1957); *Proc. Camb. Phil. Soc.* **46**:209 (1950).
30. J. S. Howland, *J. Funct. Anal.* **74**:52 (1987).
31. M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, *Phys. Rep.* **106**:121 (1984).
32. C. Piron, *Foundations of Quantum Physics* (Benjamin, Reading, Massachusetts, 1976); C. Flesia and C. Piron, *Helv. Phys. Acta* **57**:697 (1984).